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An inequality for the Selberg zeta-function, associated to the compact Riemann surface

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Abstract

We consider the absolute values of the Selberg zeta-function, associated to the compact Riemann surface, at places symmetric with respect to the line $\Re(s) = 1/2$. We prove an inequality for the Selberg zeta-function, extending the result of R. Garunkštis and A. Grigutis.

1 Introduction

Let $s = \sigma + it$ be a complex variable and $\zeta(s)$ be the Riemann zeta-function. T. S. Trudgian [6] obtained, that

$$|\zeta(1-s)| > |\zeta(s)| \tag{1}$$

except at the zeros of $\zeta(s)$, with $|t| \geqslant 6.29073$ and $\sigma > 1/2$. By well-known functional equation for the Riemann zeta-function,

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s)\zeta(1-s), \tag{2}$$

$\zeta(1-s)$ and $\zeta(s)$ have the same zeros when $0 < \sigma < 1$. Inequalities of (1) type are of great interest, since a necessary and sufficient condition for the Riemann hypothesis is $|\zeta(1-s)| > |\zeta(s)|$, where $\sigma > 1/2$ and $|t| \geqslant 6.29073$.

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Garunkštis and Grigutis examined, whether Selberg zeta-function, associated to the compact Riemann surface and the modified Selberg zeta-function satisfy the inequalities of (1) type (see Theorems 1 and 2 in [2]). Let $\mathbb{H}$ be the upper half-plane and Γ be a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Let $\Gamma \setminus \mathbb{H}$ be a compact Riemann surface of genus $g \geqslant 2$. The Selberg zeta-function, associated to the compact Riemann surface of genus $g \geqslant 2$, is defined as follows [2, 5]

$$Z_C(s) = \prod_{\{P\}} \prod_{k=0}^{\infty} (1 - N(P)^{-s-k}),$$

where $\sigma > 1$ and $\{P\}$ runs trough all the primitive hyperbolic conjugacy classes of Γ and $N(P) = \alpha^2$ if the eigenvalues of P are α and α^{-1} ($|\alpha| > 1$). The Selberg zeta-function has a meromorphic continuation to $\mathbb{C}$ [5].

The Selberg zeta-function, associated to the compact Riemann surface, satisfies (cf. (2)) [2, 4] the functional equation

$$Z_C(s) = Z_C(1-s) \exp\left(4\pi(g-1)\int_0^{s-1/2} \theta \tan \pi\theta d\theta\right). \tag{3}$$

Garunkštis and Grigutis have proved [2], that for $\sigma > 1/2$ and $t \geqslant 0.361$,

$$|Z_C(1-s)| > |Z_C(s)|.$$

In this research we extend the result of R. Garunkštis and A. Grigutis. We apply the technics of estimation used in [1].

2 An inequality for the Selberg zeta-function, associated to the compact Riemann surface

Theorem 1. *Let $Z_C(s)$ be the Selberg zeta-function associated to the compact Riemann surface of genus $g \geqslant 2$. Then, for $\sigma > 1/2$ and $t \geqslant t_0$ we have*

$$|Z_C(1-s)| > |Z_C(s)|. \tag{4}$$

Here $t_0 = 0.165...$.

By the functional equation for the Selberg zeta-function (3) we have

$$\log\left|\frac{Z_C(s)}{Z_C(1-s)}\right| = \underbrace{(g-1)}_{>0} \Re(Q(s)), \tag{5}$$

here

$$Q(s) = 4\pi \int_0^{s-1/2} v \tan \pi v \mathrm{d}v. \tag{6}$$

Integral (6) can be evaluated using triangular contour with vertices at $A(0,0)$, $B(\sigma-1/2,t)$ and $C(\sigma-1/2,0)$: $\int_{AC}+\int_{CB}+\int_{BA}=0$. Hence,

$$\underbrace{\Re(Q(s))}_{\Re\int_{AB}}=\underbrace{I_1(\sigma)}_{\int_{AC}}+\underbrace{R(\sigma,t)}_{\Re\int_{CB}}. \tag{7}$$

Here

$$I_1(\sigma)=4\pi\int_0^{\sigma-1/2}\theta\tan\pi\theta\mathrm{d}\theta \tag{8}$$

and

$$R(\sigma,t)=\Re\left\{\int_0^t i(\sigma-1/2+i\theta)\tan\left(\pi\left(\sigma-1/2+i\theta\right)\right)d\theta\right\}. \tag{9}$$

Calculating function $I_1(\sigma)$ (8), we obtain

$$I_1(\sigma)=-\frac{2}{\pi}\mathrm{Cl}_2(2\pi\sigma)-2(2\sigma-1)\log|2\sin\pi\sigma|. \tag{10}$$

Here $\mathrm{Cl}_2(x)$ is the Clausen function of order 2,

$$\mathrm{Cl}_2(x)=-\int_0^x\log\left|2\sin\frac{t}{2}\right|\mathrm{d}t. \tag{11}$$

Calculate function $R(\sigma,t)$ (9), we obtain

$$R(\sigma,t)=\underbrace{\int_0^t\frac{4\pi\theta\sin2\pi\sigma}{\cosh2\pi\theta-\cos2\pi\sigma}\mathrm{d}\theta}_{=I_2(\sigma,t)}+\underbrace{\int_0^t\frac{4\pi\left(1/2-\sigma\right)\sinh2\pi\theta}{\cosh2\pi\theta-\cos2\pi\sigma}\mathrm{d}\theta}_{=I_3(\sigma,t)} \tag{12}$$

Note that $R(\sigma,t)$ is even function by t, thus, it suffices to consider positive t values.

Calculating $I_2(\sigma,t)$, we obtain (cf. 3.531.8 in [3]), that

$$\begin{aligned}I_2(\sigma,t)=\ &4\pi\sin2\pi\sigma\int_0^t\frac{\theta}{\cosh2\pi\theta-\cos2\pi\sigma}\mathrm{d}\theta,\\ &=\frac{2}{\pi}(\Lambda(u(\sigma,t)+\pi\sigma)-\Lambda(u(\sigma,t)-\pi\sigma)-2\Lambda(\pi\sigma))=\\ &=\frac{1}{\pi}\mathrm{Cl}_2(2u(\sigma,t)+2\pi\sigma+\pi)-\frac{1}{\pi}\mathrm{Cl}_2(2u(\sigma,t)-2\pi\sigma+\pi)-\\ &-\frac{2}{\pi}\mathrm{Cl}_2(2\pi\sigma+\pi)).\end{aligned} \tag{13}$$

Here $\Lambda(x)$ is the Lobachevsky function,

$$\Lambda(x)=-\int_0^x\log|\cos t|\,\mathrm{d}t=\mathrm{Cl}_2(2x+\pi)/2+x\log2,$$

and

$$u(\sigma,t)=\arctan(\tanh\pi t\cot\pi\sigma). \tag{14}$$

Calculating $I_3(\sigma,t)$, we obtain

$$\begin{aligned} I_3(\sigma,t)=&-(2\sigma-1)\log(\cosh 2\pi t-\cos 2\pi\sigma)+\\ &+2(2\sigma-1)\log|\sin\pi\sigma|+(2\sigma-1)\log 2, \end{aligned} \tag{15}$$

Let us denote

$$L(\sigma,t)=\Re(Q(s)). \tag{16}$$

In view of (5), to prove the statement of the theorem, it is enough to show that, for $\sigma>1/2$ and $t\geqslant t_0$ the function $L(\sigma,t)$ is negative. By (7) and (10)-(15), we have

$$\begin{aligned} L(\sigma,t)=&-\frac{2}{\pi}\mathrm{Cl}_2(2\pi\sigma)+4\pi\sin 2\pi\sigma\int_0^t\frac{\theta}{\cosh 2\pi\theta-\cos 2\pi\sigma}\mathrm{d}\theta-\\ &-(2\sigma-1)\log(\cosh 2\pi t-\cos 2\pi\sigma)-(2\sigma-1)\log 2. \end{aligned} \tag{17}$$

Taking into account the duplication formula for the Clausen function,

$$\mathrm{Cl}_2(2\theta)=2\mathrm{Cl}_2(\theta)-2\mathrm{Cl}_2(\pi-\theta),$$

and the properties of the Clausen function,

$$\begin{aligned} &\mathrm{Cl}_2(\theta)=\mathrm{Cl}_2(\theta+2\pi m), \qquad m\in\mathbb{Z},\\ &\mathrm{Cl}_2(\theta)=-\mathrm{Cl}_2(-\theta). \end{aligned}$$

we obtain

$$\begin{aligned} L(\sigma,t)=&\frac{1}{\pi}\mathrm{Cl}_2(2u(\sigma,t)+2\pi\sigma+\pi)-\\ &-\frac{1}{\pi}\mathrm{Cl}_2(2u(\sigma,t)-2\pi\sigma+\pi)-\frac{1}{\pi}\mathrm{Cl}_2(4\pi\sigma)-\\ &-(2\sigma-1)\log(\cosh 2\pi t-\cos 2\pi\sigma)-(2\sigma-1)\log 2. \end{aligned} \tag{18}$$

Next we establish a lemma concerning the behaviour of the derivative of the function $L(\sigma,t)$.

Lemma 1. *For $\sigma>1/2$ and $t>0$, the derivative $L_t'(\sigma,t)$ is negative.*

Proof. Let us calculate the first partial derivative. By (7) and (12) we have

$$\begin{aligned} L_t'(\sigma,t)=&\frac{4\pi}{\cosh 2\pi t-\cos 2\pi\sigma}+\frac{4\pi(1/2-\sigma)\sinh 2\pi t}{\cosh 2\pi t-\cos 2\pi\sigma}\\ =&\underbrace{\frac{4\pi}{\cosh 2\pi t-\cos 2\pi\sigma}}_{>0}\underbrace{(t\sin 2\pi\sigma+(1/2-\sigma)\sinh 2\pi t)}_{=N(\sigma,t)}. \end{aligned} \tag{19}$$

For $\sigma \in (n-1/2, n),\ n \in \mathbb{N}$, we have $N(\sigma, t) < 0$ (since $\sin 2\pi\sigma$ is negative)

For $\sigma \in (n, n+1/2),\ n \in \mathbb{N}$,

$$N(\sigma, t) < \ t - \frac{1}{2}\sinh 2\pi t < 0.$$

For $\sigma = n,\ n \in \mathbb{N}$, we have $N(\sigma, t) = (1/2 - n)\sinh 2\pi t < 0$.

Thus, for $\sigma > 1/2$ and $t > 0$, the function $N(\sigma, t)$ is negative, yielding us (cf. (19)) $L_t^{'}(\sigma, t) < 0$. □

Let us denote

$$\begin{aligned} B_0(\sigma) =& \frac{1}{\pi}\mathrm{Cl}_2(2u(\sigma, t_0) + 2\pi\sigma + \pi) - \\ & - \frac{1}{\pi}\mathrm{Cl}_2(2u(\sigma, t_0) - 2\pi\sigma + \pi) - \frac{1}{\pi}\mathrm{Cl}_2(4\pi\sigma). \end{aligned} \tag{20}$$

Lemma 2. *The function $B_0(\sigma)$ is*

(i) *periodic with period* $P = 1$, thus $B_0(\sigma) = B_0(\sigma + m)$, $m \in \mathbb{Z}$,

(ii) *odd*, $B_0(\sigma) = -B_0(-\sigma)$,

(iii) *bounded*, $|B_0(\sigma)| \leqslant C_0$.

Here $C_0 = 0.46342...$.

Proof. The function $u(\sigma, t_0)$ (14) is periodic by σ with period $P = 1$ and the Clausen function $\mathrm{Cl}_2(\theta)$ is periodic with period $P = 2\pi$, hence the first statement of the lemma. Note that $\lim_{\sigma \to 1} B_0(\sigma) = 0$. Next, the Clausen function (of order 2) and the function $u(\sigma, t_0)$ are odd, yielding us the second statement of the lemma.

In view of (i) and (ii) it is enough to consider the function $B_0(\sigma)$ in the interval $\sigma \in (1/2, 1)$. Note that $B_0(1/2) = B_0(1) = 0$. Calculating derivatives of $B_0(\sigma)$, we obtain

$$B_0^{'}(\sigma) = -\frac{4\pi t_0 \sinh 2\pi t_0}{\cosh 2\pi t_0 - \cos 2\pi\sigma} + 2\log(\cosh 2\pi t_0 - \cos 2\pi\sigma) + 2\log 2. \tag{21}$$

and

$$B_0^{''}(\sigma) = 4\pi \underbrace{\sin 2\pi\sigma}_{<0} \underbrace{\left(\frac{2\pi t_0 \sinh 2\pi t_0}{(\cosh 2\pi t_0 - \cos 2\pi\sigma)^2} + \frac{1}{\cosh 2\pi t_0 - \cos 2\pi\sigma} \right)}_{>0}. \tag{22}$$

Let us denote

$$\omega_0(\sigma) = \frac{1}{\cosh 2\pi t_0 - \cos 2\pi\sigma}$$

and

$$a_0 = 4\pi t_0 \sinh 2\pi t_0.$$

Hence, by (21), solving the equation $B_0^{'}(\sigma_0) = 0$, we obtain

$$-a_0\omega_0 - 2\log\omega_0 + \log 4 = 0 \qquad \Longrightarrow \qquad \omega_0 = 2W(a_0)/a_0.$$

Here $W(x)$ is the Lambert W function. Thus,

$$\cos 2\pi\sigma_0 = \cosh 2\pi t_0 - \frac{2\pi t_0 \sinh 2\pi t_0}{W(4\pi t_0 \sinh 2\pi t_0)},$$

and

$$\sigma_0 = 1 - \frac{1}{2\pi}\arccos\left(\cosh 2\pi t_0 - \frac{2\pi t_0 \sinh 2\pi t_0}{W(4\pi t_0 \sinh 2\pi t_0)}\right) = 0.79336... .$$

Note that $B_0^{''}(\sigma)$ (22) is negative for $\sigma \in (1/2, 1)$, hence $B_0(\sigma_0) = 0.46342...$ is the maxima of the function, yielding us the third statement of the lemma. □

3 Proof of Theorem 1

Proof. By Lemma 1, it is enough to prove the theorem for fixed $t = t_0$. Let us denote $L_0(\sigma) = L(\sigma, t_0)$.

First consider the function $L_0(\sigma)$ for $\sigma \geqslant \sigma_1 > 1/2$. By (18) and Lemma 2, we have

$$\begin{aligned} L_0(\sigma) &< C_0 - (2\sigma - 1)\log(\cosh 2\pi t_0 - \cos 2\pi) - (2\sigma - 1)\log 2 = \\ &= C_0 - 4(\sigma - 1/2)\log(2\sinh \pi t_0). \end{aligned} \tag{23}$$

Hence, $L_0(\sigma) < 0$ for $\sigma \geqslant \sigma_1$. Here

$$\sigma_1 = \frac{1}{2} + \frac{C_0}{4\log(2\sinh \pi t_0)} = 1.94001....$$

Next, consider the function $L_0(\sigma)$ for $1/2 < \sigma < \sigma_1$. Calculating derivatives of the function, we obtain (cf. (18), (20) and (21)), that

$$\begin{aligned} L_0^{'}(\sigma) &= \underbrace{\frac{-4\pi}{\cosh 2\pi t_0 - \cos 2\pi\sigma}}_{<0} \underbrace{(d_0 + (\sigma - 1/2)\sin 2\pi\sigma)}_{=f_0(\sigma)}, \\ L_0^{''}(\sigma) &= -4\pi \frac{\sin 2\pi\sigma + 2\pi(\sigma - 1/2)\cos 2\pi\sigma}{\cosh 2\pi t_0 - \cos 2\pi\sigma} + \\ &\quad + \frac{8\pi^2 \sin 2\pi\sigma}{(\cosh 2\pi t_0 - \cos 2\pi\sigma)^2} f_0(\sigma). \end{aligned} \tag{24}$$

Here $d_0 = t_0 \sinh 2\pi t_0 = 0.203...$.

The function $f_0(\sigma)$ has three zeros in the interval $(1/2, \sigma_1)$.

Indeed, for $\sigma \in (1/2, 3/4)$, the derivative $f_0'(\sigma)$ is negative, while $f_0(1/2) > 0$ and $f_0(3/4) < 0$. Hence, the first root $\hat{\sigma}_1 \in (1/2, 3/4)$. Note that $L_0''(\hat{\sigma}_1) > 0$ (cf.(24)).

For $\sigma \in (3/4, 1)$, the second derivative $f_0''(\sigma)$ is positive (hence the function is convex in the interval), while $f_0(3/4) < 0$ and $f_0(1) > 0$. Thus, the second root $\hat{\sigma}_2 \in (3/4, 1)$. Calculating numerically, we obtain $\hat{\sigma}_2 = 0.919...$ and $L_0''(\hat{\sigma}_2) < 0$.

For $\sigma \in (1, 3/2)$, the function $f_0(\sigma)$ is positive, hence no zeros in the interval.

For $\sigma \in (3/2, 7/4)$, the derivative $f_0'(\sigma)$ is negative, while $f_0(3/2) > 0$ and $f_0(7/4) < 0$. Hence, the third root $\hat{\sigma}_3 \in (3/2, 7/4)$. Note that $L_0''(\hat{\sigma}_3) > 0$ (cf.(24)).

For $\sigma \in (7/4, \sigma_1)$, the second derivative $f_0''(\sigma)$ is positive (hence the function is convex in the interval), while $f_0(7/4) < 0$ and $f_0(\sigma_1) < 0$, hence no zeros in the interval.

The only maxima of the function $L_0(\sigma)$ for $1/2 < \sigma < \sigma_1$ corresponds $\hat{\sigma}_2$. However, $L_0(\hat{\sigma}_2)$ is negative, yielding us the statement of the theorem. $\square$

Remark 1. Let us denote for $\sigma \in (n + 1/2, n + 1)$, $n \in \mathbb{N}_0$,

$$\varphi_n(\sigma) = \{t | L(\sigma, t) = 0\}.$$

Then

$$t_0 = \max_{1/2<\sigma<1} \varphi_0(\sigma).$$

Calculating t_0 numerically, we obtain optimal $t_0 = 0.165...$.

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